ESTIMATION AND LOCATION USING MULTIPLE INFRASOUND ARRAYS

Robert H. Shumway
Sung-Eun Kim
University of California, Davis

Defense Threat Reduction Agency, DOD
Contract DSWA01-97-0150

ABSTRACT

Integrating or *fusing* array data from various sources will be extremely important in making the best use of networks for detecting infrasonic signals and for estimating their velocities and azimuths. In addition, studying the size and shape of location ellipses that use velocity, azimuth and travel time information from an integrated collection of small arrays to locate the event will be critical in evaluating our overall capability for monitoring a CTBT.

In the first phase of this study, we have developed a small-array theory that characterizes the uncertainty in estimated velocities and azimuths for different infrasonic array configurations and levels of signal correlation. The performance of simple beam forming and a generalized likelihood beam that is optimal under signal correlation have been compared. Empirical work to establish plausible signal frequencies and coherence levels for small arrays has concentrated on several events in the Pacific.

The second phase of the study develops statistical methods for integrating information from a collection of small arrays in order to obtain unified estimators for the location of an event as well as the predictive region that best characterizes location uncertainty. We derive optimal methods based on a nonlinear Bayesian approach to combining wavenumber parameters and their uncertainties into an uncertainty ellipse for the location. The methodology is capable of combining uncertainties for the overall input variances with observed data from small arrays to get posterior predictive regions for locations. Several contrived examples are used to compute ellipses that might be expected from a typical set of small infrasonic arrays.

Key Words: *Detection, Location, Infrasound, Multiple array processing, Nonlinear estimation, Bayesian methods*

OBJECTIVES

The first phase of this study focused on completely characterizing the statistical detection and estimation capabilities of small infrasonic arrays, as contemplated by the Prototype International Data Centre (PIDC). For a multivariate correlated signal model, we developed the statistical approaches for (i) maximum likelihood detection, (ii) estimation of velocity and azimuth parameters, (iii) estimation of noise spectra and the signal spectral matrix and (iv) performance of various array configurations. Applications of the results to several events, coupled with theoretical calculations based on the Mack-Flinn distance-coherence model, gave predicted azimuthal standard deviations ranging from 1 to 5 degrees, depending on the center frequency of the signal, array configuration and signal to noise ratio. Summaries of the results obtained can be found in Shumway and Kim (1998) and in Shumway et al (1999).

In the second phase of this study, we are examining the implications that the results of the first phase have for location uncertainty using estimated wave-number parameters from more than one array. In general, there may be a small number of arrays that detect any given event and we look at different methods for estimating the uncertainty of locations derived from these kinds of data. The results take wave-number coordinates from multiple arrays and combine or *fuse* them into an overall location for the common event and obtain an associated uncertainty region for the location. As has been observed by numerous authors, the assumptions made about the variances of the observed errors in estimated wave-numbers or travel times will influence the size of the uncertainty region. In this analysis, we focus on three assumptions regarding the unknown variances, namely that they are (A) known exactly (Evernden, 1969), (B) completely unknown (Flinn, 1965) and (C) subject to a prior distribution (Jordan and Sverdrup, 1981, Bratt and Bache, 1988). We give several examples involving small arrays and an event location similar to what might be expected from a subset of arrays recording an event on the proposed PIDC network of infrasonic stations .

RESEARCH ACCOMPLISHED

MODEL DEVELOPMENT

In the case of an infrasonic array, it is convenient to assume that observed data at sensor $j = 1, 2, \ldots, N$ for array $k = 1, 2, \ldots, n$ records

$$y_{jk}(t) = s_{jk}(t - T_{jk}) + n_{jk}(t) \tag{1}$$

as signal plus noise, where the time delays

$$T_{jk}(\boldsymbol{\theta}) = -\frac{\boldsymbol{r}'_{jk}\boldsymbol{\theta}_k}{\nu} \tag{2}$$

depend on the array coordinates $\boldsymbol{r}_{jk} = (r_{jk1}, r_{jk2})'$ and a wave-number vector $\boldsymbol{\theta}_k = (\theta_{1k}, \theta_{2k})'$ and ν is the center frequency. It should be noted that the wave-numbers can be related to the velocity and azimuth of the propagating signal that is of interest (see Shumway et al, 1999) . We will not express the equations here in those terms but will simply note that $\boldsymbol{\theta}_k = \boldsymbol{\theta}_k(\boldsymbol{x})$ is related to the event location $\boldsymbol{x} = (x_1, x_2)'$ through

$$\boldsymbol{\theta}_k(\boldsymbol{x}) = \frac{\nu}{V}\frac{(\boldsymbol{c}_k - \boldsymbol{x})}{d_k(\boldsymbol{x})} \tag{3}$$

where $\boldsymbol{c}_k = (c_{1k}, c_{2k})'$ denotes the coordinates of the center of the k^{th} array and V is the velocity of the signal with ν and V generally taken as being fixed and known. The signs of the two components of $\boldsymbol{\theta}_k(\boldsymbol{x})$

change, depending on where the event is located relative to the center coordinates of the array. The distance term in the denominator is obviously also a function of the location, given by

$$d_k(\boldsymbol{x}) = ||\boldsymbol{c}_k - \boldsymbol{x}||. \tag{4}$$

There will usually be estimators available for the n wave-number vectors, say $\hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2, \ldots, \hat{\boldsymbol{\theta}}_n$, available from each of the n arrays, along with the asymptotic covariance matrices, say $\Sigma_k, k = 1, \ldots, n$. These have been determined in Shumway et al (1999) for both beam-forming and correlated beam-forming frequency wave-number estimators.

In order to give a concrete example, consider the three arrays in Figure 1, with centers at $\boldsymbol{c}_1 = (0,0)', \boldsymbol{c}_2 = (0,-1000)', \boldsymbol{c}_2 = (-1000,0)'$, observing an hypothetical event at $\boldsymbol{x} = (-2366, -2366)'$, with all distances measured in km.

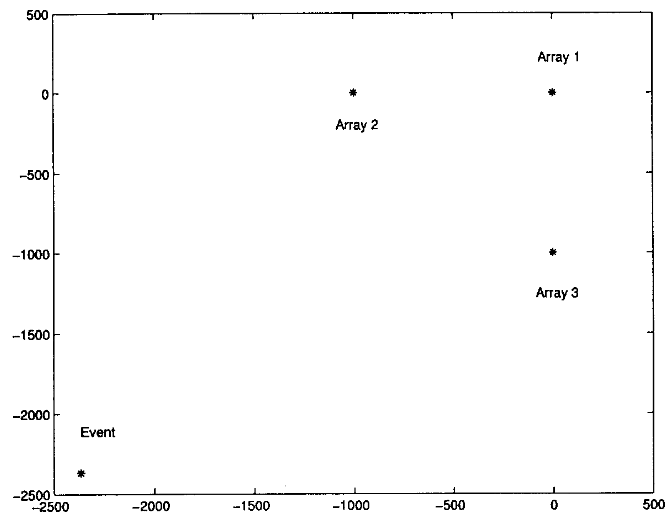

Figure 1: Hypothetical array centers for location example

Each of the arrays, centered at their respective values, will generate an estimator for the wave-number parameters relative to the event. Since this particular event is in the third quadrant for all of the arrays the coefficients of the two components of (3) will all be negative. In this case, the location is $\boldsymbol{x} = (-2366, -2366)'$ and the true wave-numbers become are $\boldsymbol{\theta}_1(\boldsymbol{x}) = (-.0733, -.1270)'$, $\boldsymbol{\theta}_2(\boldsymbol{x}) = (-.1037, -.1037)'$ and $\boldsymbol{\theta}_3(\boldsymbol{x}) = (-.1270, -.0733)'$, as can be computed from (3), with $\nu = .044$ Hz and $V = .3$ km/sec.

The above discussion suggests that we might assume a model for the estimated wave-number vectors that takes the form

$$\hat{\boldsymbol{\theta}}_k = \boldsymbol{\theta}_k(\boldsymbol{x}) + \boldsymbol{e}_k \tag{5}$$

where the errors are assumed to have 2×2 covariance matrices of the form

$$\text{cov } \boldsymbol{e}_k = \sigma^2 \Sigma_k, \tag{6}$$

and σ^2 denotes the variance parameter. The problem is to estimate the location vector, given a sample of 2×1 wave-number estimators $\hat{\boldsymbol{\theta}}_1, \ldots, \hat{\boldsymbol{\theta}}_n$ and the covariance matrices $\Sigma_1, \ldots, \Sigma_n$. The model (5) is is nonlinear in the location parameter vector $\boldsymbol{x}$ because of (3) and (4).

It should be noted, at this point, that travel times may also be known from considerations apart from the wave-number estimation methodology discussed in Shumway et al (1999). For example, travel times may also be determined by cross correlating elements in the array, using onset times read by an analyst or monitoring the ratio of short-term to long-term mean squares. While such information cannot be completely independent of the wavenumber analysis, it may present a more global solution in the sense that the observations do not depend on focusing on a given frequency band. If such estimated travel times are available, the model in (5) might be augmented by one expressing the estimated travel times , say $\hat{t}_1, \ldots, \hat{t}_n$ in a form analgous to (5) where

$$t_k(\boldsymbol{x}) = \frac{d_k(\boldsymbol{x})}{V} \tag{7}$$

is again a nonlinear function of the location vector $\boldsymbol{x}$.

Wave-number Estimation and Uncertainty

Shumway et al (1999) have investigated optimal detection and estimation of the wave-number vector and its uncertainty. They develop the F-statistic for detection under perfect signal correlation and a generalized beam for detection when the signal correlation degrades with distance. Under the model given in (1) and (2), they obtain the maximum likelihood estimators of the wave-number vector, say $\hat{\boldsymbol{\theta}}$. The asymptotic covariance matrix, in the case of beamforming and perfect signal correlation, is given by

$$\Sigma \approx \frac{1}{2(2\pi)^2} \frac{1}{L} \frac{r}{N} \left(1 + \frac{r}{N}\right) R^{-1} \tag{8}$$

where N is the array size, L is the number of frequencies smoothed for the detector (time-bandwidth product), R is the covariance matrix of the array coordinates and r is inverse of the signal to noise ratio.

The estimated wave-numbers from n arrays, say $\hat{\boldsymbol{\theta}}_1, \ldots, \hat{\boldsymbol{\theta}}_n$ and their asymptotic covariance matrices, $\Sigma_1, \ldots, \Sigma_n$ serve as natural inputs for the fused locations given in the next section.

Fused Locations: Classical and Bayesian Methods

We extend the classical methods first to the case where we observe wave-number parameters and their covariance matrix from n arrays and wish to combine or *fuse* the information into an overall lcoation. The nonlinear model (5) can be treated in the usual way. That is, expand $\boldsymbol{\theta}_k(\boldsymbol{x})$ around some initial value, say $\boldsymbol{x} = \boldsymbol{x}_0$ and write a linearization as

$$\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k(\boldsymbol{x}_0) = A_k(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{e}_k, \tag{9}$$

where

$$A_k(\boldsymbol{x}) = \frac{\partial \boldsymbol{\theta}_k(\boldsymbol{x})}{\partial \boldsymbol{x}} \tag{10}$$

is the usual 2×2 matrix of partial derivatives of $\boldsymbol{\theta}_k(\boldsymbol{x})$. Then, stacking the n, 2×1 wave-number vectors and minimizing the weighted sum of squared errors can be done by sucessively estimating $\boldsymbol{\beta} = \boldsymbol{x} - \boldsymbol{x}_0$. This leads to

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}_0 + C^{-1}(\boldsymbol{x}_0) \sum_{k=1}^{n} A_k(\boldsymbol{x}_0)' \, \Sigma_k^{-1} [\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k(\boldsymbol{x}_0)], \tag{11}$$

where

$$C(\boldsymbol{x}_0) = \sum_{k=1}^{n} A_k(\boldsymbol{x}_0)' \ \Sigma_k^{-1} A_k(\boldsymbol{x}_0). \tag{12}$$

It follows that the approximate covariance matrix of the final estimator is

$$\text{cov } \hat{\boldsymbol{x}} = \sigma^2 C^{-1}(\hat{\boldsymbol{x}}). \tag{13}$$

Equations (10) and (11) exhibit the *fusion estimators* at each stage as pooled estimators over the n arrays as long as the variances are known. We may also develop a confidence ellipse for the fusion estimators under assumptions (A), (B) and (C) mentioned earlier.

(A) **Variance Known:** We may assume that the variance σ^2 is known, either from the statistical variances of the computed wave-number estimators or from a combination of factors including the statistical wave-number variances. In this case, the generalization of the usual chi-squared ellipse considered by Evernden (1969) can be computed from the fact that

$$(\boldsymbol{x} - \hat{\boldsymbol{x}})' C(\hat{\boldsymbol{x}}) (\boldsymbol{x} - \hat{\boldsymbol{x}}) \sim \sigma^2 \chi_2^2, \tag{14}$$

where $\sim$ denotes *is distributed as* and χ_2^2 denotes a chi-squared distribution with 2 degrees of freedom. Note that the statistical uncertainty of the wave-number estimators is already in the matrix Σ_k so that a plausible estimator for σ^2 in the absence of other factors might by unity.

(B) **Variance Unknown:** If variances are known only up to the constant σ^2, it must be estimated. are completely unknown and must be determined from the small-array data. For the Gaussian case, the maximum likelihood estimator is proportional to the unbiased estimator

$$s^2 = \frac{1}{2(n-1)} \sum_{k=1}^{n} (\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k(\hat{\boldsymbol{x}}))' \ \Sigma_k^{-1} (\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k(\hat{\boldsymbol{x}})). \tag{15}$$

This case, originally considered in Flinn(1965), leads to a confidence interval based on the F-distribution, namely

$$(\boldsymbol{x} - \hat{\boldsymbol{x}})' C(\hat{\boldsymbol{x}}) (\boldsymbol{x} - \hat{\boldsymbol{x}}) \sim 2s^2 F_{2,2(n-1)}, \tag{16}$$

where $F_{2,2(n-1)}$ denotes the F-distribution with 2 and $2(n-1)$ degrees of freedom.

(C) **Variance Subject to Prior Distribution:** It is often the case that it is unrealistic to assume that the variance is known exactly because (14) becomes too small. For a small number of arrays, the ellipse based on the F-statistic (16) is often much too large. A useful compromise, introduced by Jordan and Sverdrup (1981)and continued by Bratt and Bache (1988), is to quantify the initial uncertainty about σ^2 by assigning it a prior distribution with density function $\pi(\sigma^2)$. It is convenient to use the inverted chi-squared distribution with parameters m, representing the equivalent sample size embodied in the prior information and σ_0^2, representing a prior centering value for the variance. For the form of the density function, see Anderson (1984). Figure 2 plots the density function for the standard deviation σ for $\sigma_0 = .02$ and $m = 10, 30$. We note that the two values put the standard deviation between .01 and .05 for $m = 10$ and between .015 and .03 for $m = 30$ For a fully Bayesian approach, we assume a non-informative prior on $(-\infty < x_1, x_2 < \infty)$ for the location $\boldsymbol{x}$ and compute the posterior distribution, given the wave-number observations, as a bivariate t-distribution with 2 and $2(n-1)+m$ degrees of freedom. The posterior estimator for the variance is

$$\sigma'^2 = \frac{2(n-1)s^2 + m\sigma_0^2}{2(n-1)+m} \tag{17}$$

implying that the best approach is simply to pool the initial variance σ_0^2 and the sample variance s^2, weighted by their degrees of freedom. The quadratic form involving the location vector $\boldsymbol{x}$ in the multivariate t has an F-distribution, making the 95% posterior probability ellipse for the location expressible as

$$(\boldsymbol{x} - \hat{\boldsymbol{x}})' C(\hat{\boldsymbol{x}})(\boldsymbol{x} - \hat{\boldsymbol{x}}) \sim 2\sigma'^2 F_{2,2(n-1)+m} \tag{18}$$

It is interesting that the form of the posterior probability ellipse (18) is similar to (16) but will be tighter because of the additional degrees of freedom for the F-statistic. Hence, the Bayesian solution represents a compromise between (14) and (16), the methods of (A) and (B).

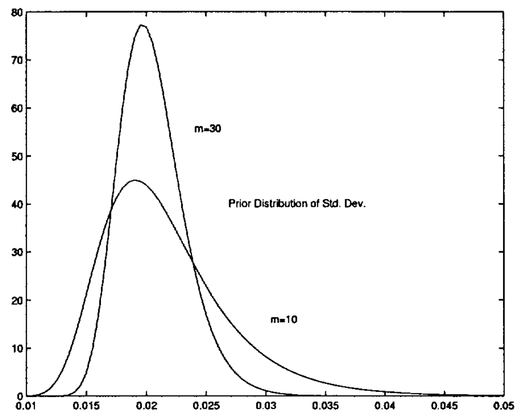

Figure 2: Possible prior distributions for standard deviation of wavenumber estimates

Preliminary Data Analysis

We illustrate some of the potential of the equations derived in the previous section with the contrived array given in Figure 1. We consider two possible configurations of recording arrays, namely all three arrays recording or only the two arrays centered at $\boldsymbol{c}_1 = (0,0)'$ and $\boldsymbol{c}_3 = (0,-1000)'$ recording. This gives a comparison between a wider and more narrow aperture.

The known location of the event in question was taken to be $\boldsymbol{x} = (-2360, -2360)$, as shown in Figure 1. The wavenumber estimates were simulated by computing $\boldsymbol{\theta}_k$ form (3) and adding random noise, according to (5), with the covariance matrix as computed from (8) . The array was assumed to have $N = 4$ elements arranged in a triangle with 1 km sides and a center element. The signal to noise ratio was taken as 3 and the degrees of freedom as $L = 17$. Locations were computed using the random inputs and a covariance matrix computed from (8). Ten Gauss-Newton iterations were performed using the approach in (9)-(12)

The overall performance of subsets of arrays recording various events within this region is being studied using simulated data and reasonable expected performance parameters for infrasonic arrays

Finally, it has been found that the locations are susceptible to perturbations in the derivatives involved in the nonlinear methodology implied by Equations (9)-(12). The Bayesian approach may be helpful in this context. Note that, in the Bayesian approach, we assume that $\hat{\boldsymbol{\theta}}_k$ is normally distributed with mean $\boldsymbol{\theta}_k(\boldsymbol{x})$ where the distribution is conditional on $\boldsymbol{x}$ and σ^2, which are jointly distributed as a uniform and inverted chi-squared distribution respectively. The posterior distribution of the location $\boldsymbol{x}$, conditional on the data $\hat{\boldsymbol{\theta}}$ will be of the form

$$P(\boldsymbol{x}|\hat{\boldsymbol{\theta}}_1, \ldots, \hat{\boldsymbol{\theta}}_n) \propto [Q(\boldsymbol{x})]^{-(m+2n)/2} \tag{19}$$

where

$$Q(\boldsymbol{x}) = \sum_{k=1}^{n} \big(\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k(\boldsymbol{x})\big)' \Sigma_k^{-1} \big(\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k(\boldsymbol{x})\big) \tag{20}$$

Then, whether (19) is regarded as a posterior distribution in the Bayesian sense or an integrated likelihood, in the sense of Berger et al (1999), we seek $\hat{x}$ as the minimizer of the quadratic form $Q(\boldsymbol{x})$. This is exactly the objective function considered in the conventional approach using (11). Expanding $Q(\boldsymbol{x})$ about the resulting minimizer $\hat{\boldsymbol{x}}$ leads to the same solution. However, it may be possible to develop a derivative-free approach to treating the nonlinearity of $\hat{\theta}_k(\boldsymbol{x})$ using resampling techniques and we are looking at this possibility in the sequel.

to get the final estimated locations, which turned out to be $\hat{x} = (-2330, -2344)'$ for the full array and $\hat{x} = (-2233, -2282)'$ for the reduced array.

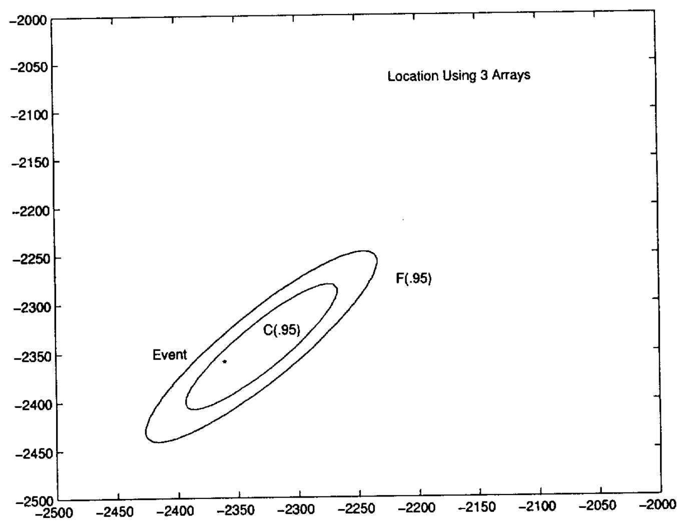

Figure 3: Location of hypothetical event at (-2366,-2366) using all three arrays in Figure 1

Figures 3 and 4 show the confidence ellipses, computed by methods (A) and (B), with the smaller ellipse being the chi-squared ellipse with known variance. Depending on the prior assumptions as to the distribution of the variance, the Bayesian posterior probability ellipse would like somewhere between the chi-squared and F ellipses. The contrived example gives an idea as to the relative sizes of the ellipses expected under different variance assumptions and under the different array configurations. Note that the case where only two arrays recorded gives a much larger ellipse and that both ellipses reflect the fact that the observing arrays had a relatively narrow aperture (30 degrees for $n = 3$ and 15 degrees for $n = 2$).

CONCLUSIONS AND RECOMMENDATIONS

The detection and location of events depends on a number of parameters that can be varied to develop plausible performance measures for different configurations of recording arrays and event locations . Optimum detection and estimation of wave-number parameters, including velocity and azimuth, and the uncertainty of such estimators has been covered during the first year of this contract and appears in Shumway et al (1999). In the present phase of the study , we have developed expressions for the location and its uncertainty using an optimum fusion of wave-number parameters from an arbitrary configuration of recording arrays. These estimators and uncertainty regions, given in the previous section, allow us to compute the expected performance as a function of the configurations of both the small recording arrays and the geometry of the multiple arrays recording the event.

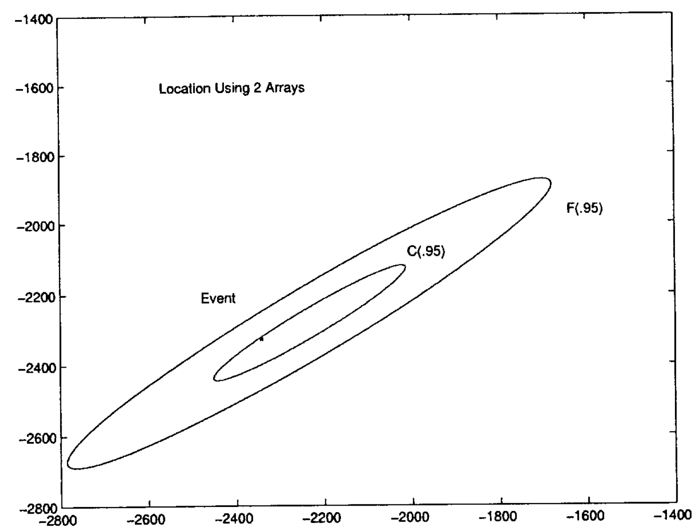

Figure 4: Location of hypothetical event at (-2366,-2366) using two arrays (0,0) and (0,-1000) Figure 1

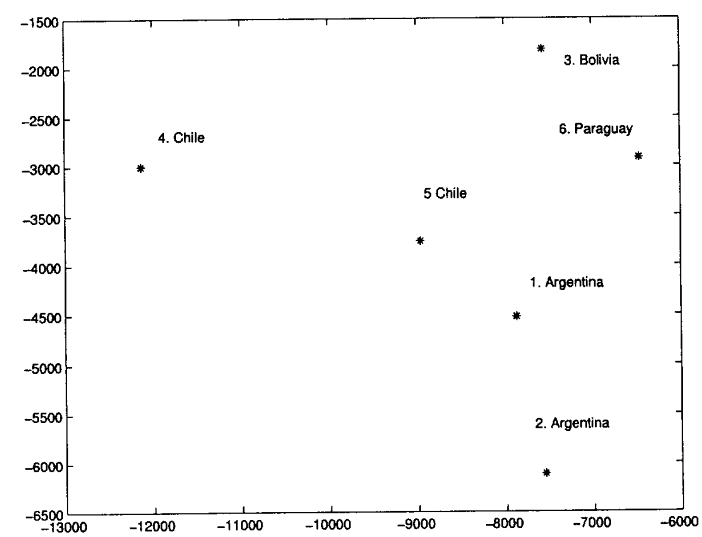

Figure 5: A subset of infrasonic arrays planned for the Prototype International Data Center

However, our primary focus is to characterize the location performance of combinations of infrasonic arrays in the PIDC network. Figure 5 shows a small area and the locations of arrays planned for that area.

REFERENCES

Anderson, T.W. (1984). *An Introduction to Multivariate Statistical Analysis.* New York: Wiley.

Berger, J.O, B. Liseo and R.L. Wolpert (1999). Integrated likelihood methods for eliminating nuisance parameters. *Statistical Science*, **14**, 1-28.

Bratt, S.R. and T.C. Bache (1988). Locating events with a sparse network of regional arrays. *Bull. Seismolog. Soc. Amer*, **78**, 780-798.

Evernden, J.F. (1969). Precision of epicenters obtained by small numbers of world-wide stations. *Bull Seismolog. Soc. Amer.*, **59**, 1365-1398.

Flinn, E.A. (1965). Confidence regions and error determinations for seismic event location. *Rev. Geophys.*, **3**, 157-185.

Jordan, T.H. and K.A. Sverdrup (1981). Teleseismic location techniques and their application to earthquake clusters in the South-Central Pacific, *Bull Seismolog. Soc. Amer*, **71**, 1105-1130.

Shumway, R.H., S.E. Kim and R.R. Blandford (1999). Nonlinear estimation for time series observed on arrays. Chapter 7, Ghosh ed. *Asymptotics, Nonparametrics and Time Series*, 227-258. New York: Marcel Dekker.

Shumway, R.H. and S.E. Kim (1998). Detection and estimation on small infrasonic arrays. In *Proceedings of 20th Annual Seismic Research Symposium on Monitoring a Comprehensive Test Ban Treaty (CTBT)*, ed. J. Fantroy D. Heatley, J. Warren, F. Chavez and C. Meade, Dept. of Defense, Nuclear Treaty Programs and U.S. Dept. of Energy, Office of Nonproliferation and National Security.